Exercise 20

Find the intersection of the planes $x + 2y + z = 0$ and $x - 3y - z = 0$.

Solution

The intersection for two planes is a straight line, which can be parameterized as

$$\mathbf{y}(t) = \mathbf{m}t + \mathbf{b},$$

where $\mathbf{m}$ is the direction vector and $\mathbf{b}$ is the position vector for any point on the line. The normal vectors to the given planes are obtained from the coefficients of x, y, and z: $(1, 2, 1)$ and $(1, -3, -1)$. Take the cross product of these two to find the direction vector of the line.

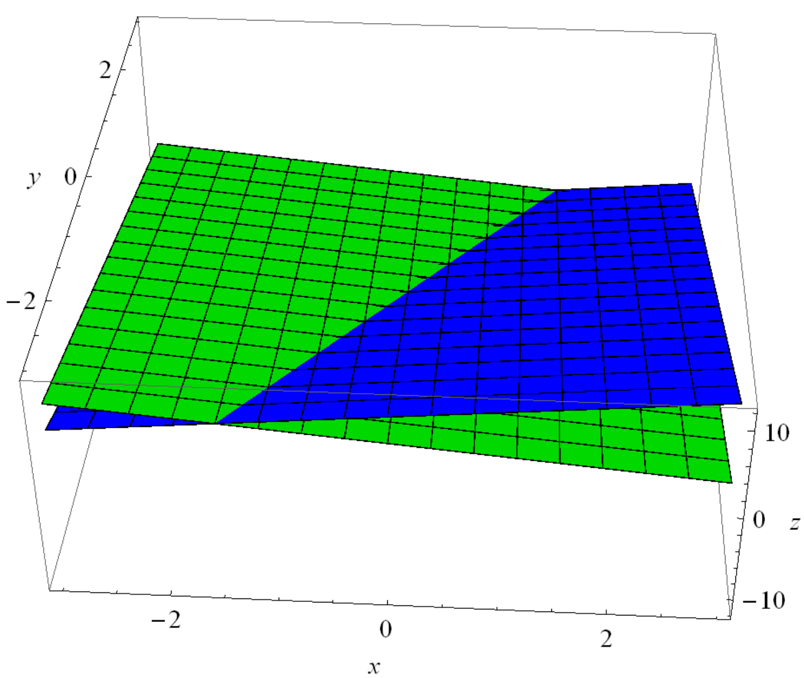
$$\mathbf{m} = (1,2,1)\times(1,-3,-1) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 2 & 1 \\ 1 & -3 & -1 \end{vmatrix} = (-2+3)\hat{\mathbf{x}} - (-1-1)\hat{\mathbf{y}} + (-3-2)\hat{\mathbf{z}} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} - 5\hat{\mathbf{z}} = (1, 2, -5)$$

All that's left is to find a point common to both planes.

$$\left.\begin{aligned} x + 2y + z = 0 \\ x - 3y - z = 0 \end{aligned}\right\}$$

Choose $x = 0$, $y = 0$, and $z = 0$, for example. Then $\mathbf{b} = (0, 0, 0)$, and the line is

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{m}t + \mathbf{b} \\ &= (1, 2, -5)t + (0, 0, 0) \\ &= (t, 2t, -5t). \end{aligned}$$



In green is $x + 2y + z = 0$, and in blue is $x - 3y - z = 0$.